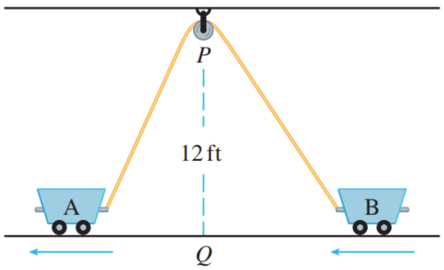
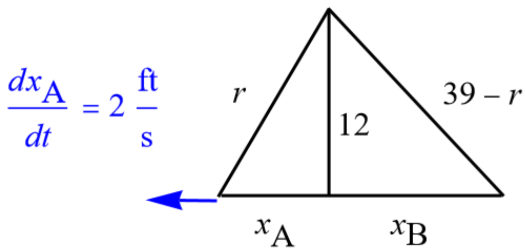
Exercise 42

Two carts, A and B, are connected by a rope 39 ft long that passes over a pulley P (see the figure). The point Q is on the floor 12 ft directly beneath P and between the carts. Cart A is being pulled away from Q at a speed of 2 ft/s. How fast is cart B moving toward Q at the instant when cart A is 5 ft from Q?



Solution

Draw a schematic of the carts at a certain time.



The aim is to find dx_B/dt when $x_A = 5$. There are two right triangles, and the Pythagorean theorem holds for each of them.

$$\left.\begin{aligned} r^2 &= x_A^2 + 12^2 \\ (39 - r)^2 &= x_B^2 + 12^2 \end{aligned}\right\}$$

Subtract the respective sides of the first equation from those of the second equation.

$$(39 - r)^2 - r^2 = (x_B^2 + 12^2) - (x_A^2 + 12^2)$$

$$(1521 - 78r + r^2) - r^2 = x_B^2 - x_A^2$$

$$1521 - 78r = x_B^2 - x_A^2$$

Solve for x_B.

$$x_B^2 = x_A^2 - 78r + 1521$$

$$x_B = \sqrt{x_A^2 - 78r + 1521}$$

$$= \sqrt{x_A^2 - 78\sqrt{x_A^2 + 12^2} + 1521}$$

As a result,

$$x_\text{B} = \sqrt{x_\text{A}^2 - 78\sqrt{x_\text{A}^2 + 144} + 1521}.$$

Take the derivative of both sides with respect to time by using the chain rule repeatedly.

$$\frac{d}{dt}(x_\text{B}) = \frac{d}{dt}\left(\sqrt{x_\text{A}^2 - 78\sqrt{x_\text{A}^2 + 144} + 1521}\right)$$

$$\begin{aligned}
\frac{dx_\text{B}}{dt} &= \frac{1}{2}\left(x_\text{A}^2 - 78\sqrt{x_\text{A}^2 + 144} + 1521\right)^{-1/2} \cdot \frac{d}{dt}\left(x_\text{A}^2 - 78\sqrt{x_\text{A}^2 + 144} + 1521\right) \\
&= \frac{1}{2}\left(x_\text{A}^2 - 78\sqrt{x_\text{A}^2 + 144} + 1521\right)^{-1/2} \cdot \left[(2x_\text{A}) \cdot \frac{dx_\text{A}}{dt} - \frac{78}{2}\left(x_\text{A}^2 + 144\right)^{-1/2} \cdot \frac{d}{dt}\left(x_\text{A}^2 + 144\right)\right] \\
&= \frac{1}{2}\left(x_\text{A}^2 - 78\sqrt{x_\text{A}^2 + 144} + 1521\right)^{-1/2} \cdot \left[(2x_\text{A}) \cdot \frac{dx_\text{A}}{dt} - \frac{78}{2}\left(x_\text{A}^2 + 144\right)^{-1/2} \cdot \left(2x_\text{A} \cdot \frac{dx_\text{A}}{dt}\right)\right] \\
&= \frac{1}{2}\left(x_\text{A}^2 - 78\sqrt{x_\text{A}^2 + 144} + 1521\right)^{-1/2} \cdot \left(2x_\text{A}\frac{dx_\text{A}}{dt} - \frac{78x_\text{A}}{\sqrt{x_\text{A}^2 + 144}}\frac{dx_\text{A}}{dt}\right) \\
&= x_\text{A}\left(x_\text{A}^2 - 78\sqrt{x_\text{A}^2 + 144} + 1521\right)^{-1/2} \cdot \left(1 - \frac{39}{\sqrt{x_\text{A}^2 + 144}}\right)\frac{dx_\text{A}}{dt} \\
&= \frac{x_\text{A}}{\sqrt{x_\text{A}^2 - 78\sqrt{x_\text{A}^2 + 144} + 1521}} \cdot \left(1 - \frac{39}{\sqrt{x_\text{A}^2 + 144}}\right)(2)
\end{aligned}$$

At the instant that cart A is 5 feet from Q, the rate of change of x_B with respect to time is

$$\left.\frac{dx_\text{B}}{dt}\right|_{x_\text{A}=5} = \frac{(5)}{\sqrt{(5)^2 - 78\sqrt{(5)^2 + 144} + 1521}} \cdot \left(1 - \frac{39}{\sqrt{(5)^2 + 144}}\right)(2) = -\frac{10}{\sqrt{133}}\ \frac{\text{ft}}{\text{s}} \approx -0.86711\ \frac{\text{ft}}{\text{s}}.$$

The minus sign indicates that the distance x_B gets smaller as time goes on. Therefore, when cart A is 5 feet from Q, cart B is moving toward Q at a speed of about 0.86711 feet per second.